Section 2.5 – Zeros of Polynomial Functions

The polynomial function $f(x)=63x^3-129x^2+28x+20$ has $x=\frac{5}{3},-\frac{2}{7},\frac{2}{3}$ as its zeros.

Notice that the NUMERATORS of these zeros (5, -2, and 2) are factors of the *constant term*, 20.

Also notice that the DENOMINATORS (3 and 7) are factors of the *leading coefficient*, 63.

The Rational Zero Theorem

If $f(x)=a_n x^n+\ldots+a_1 x^1+a_0$ has integer coefficients, then *every* rational zero of $f(x)$ has the following form:

$$\frac{p}{q}=\frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Ex 1) Find the rational zeros of $f(x)=x^3+2x^2-11x-12$

1st: List the possible rational zeros

constant term's factors: ±1, ±2, ±3, ±4, ±6, ±12

leading coefficient's factors: ±1

So, the possible rational zeros are:

$x = \frac{\pm 1}{1}, \frac{\pm 2}{1}, \frac{\pm 3}{1}, \frac{\pm 4}{1}, \frac{\pm 6}{1}, \frac{\pm 12}{1}$ $x = -12, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 12$

2nd: Test these zeros using synthetic division

Try x = 1

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -11 & -12 \\ & & 1 & 3 & -8 \\ \hline & 1 & 3 & -8 & -20 \end{array}$$

No!

Try x = -1

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -11 & -12 \\ & & -1 & -1 & 12 \\ \hline & 1 & 1 & -12 & 0 \end{array}$$

Yes!

Since −1 is a zero of $f(x)$, you can write the following: $f(x) = (x+1)(x^2+x-12)$

Factor the trinomial and write the function in fully factored form:

$f(x) = (x+1)(x+4)(x-3)$ so the zeros are: $x = -1, -4, 3$

Section 2.5 – Zeros of Polynomial Functions

Ex 2) Find the rational zeros of $f(x)=2x^3-3x^2-8x-3$

const: $\pm 3, \pm 1$
coef: $\pm 1, \pm 2$

1st: List the possible rational zeros

The possible rational zeros are:

$x = \pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{1}{2}, \pm\frac{1}{1} \rightarrow$ $x = -3, -\frac{3}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{3}{2}, 3$

2nd: Test these zeros using synthetic division

$$
\begin{array}{r|rrrr}
3 & 2 & -3 & -8 & -3 \\
 & & 6 & 9 & 3 \\
\hline
 & 2 & 3 & 1 & 0
\end{array}
$$

yes!

$f(x) = (x-3)(2x^2+3x+1)$

Factor the trinomial and write the function in fully factored form:

$f(x) = (x-3)(2x+1)(x+1)$ so the zeros are: $x = 3, -\frac{1}{2}, -1$

Section 2.5 – Zeros of Polynomial Functions

Ex 3) Use the following function $f(x)=2x^3+2x^2-8x-8$ to:

Find possible zeros, pick some and test them using synthetic division, factor completely. Then state all zeros.

p: ±8, ±4, ±2, ±1 > ±8, ±4, ±2, ±1, ±1/2
q: ±2, ±1

$$\begin{array}{r|rrrr} 2 & 2 & 2 & -8 & -8 \\ & & 4 & 12 & 8 \\ \hline & 2 & 6 & 4 & 0 \end{array}$$ Yes!

$(x-2)(2x^2+6x+4)=(x-2)(2x+2)(x+2)$

$x=2,-1,-2$

Ex 4) Use the following function $f(x)=x^4-x^3+x^2-3x-6$ to:

Find possible zeros, pick some and test them using synthetic division, factor completely. Then state all zeros.

p: ±6, ±3, ±2, ±1 > ±6, ±3, ±2, ±1
q: ±1

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 1 & -3 & -6 \\ & & -1 & 2 & -3 & 6 \\ \hline & 1 & -2 & 3 & -6 & 0 \end{array}$$ Yes!

$(x+1)(x^3-2x^2+3x-6)$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 0 \end{array}$$ Yes!

$(x+1)(x-2)(x^2+3)$

Zeros: $-1, 2, \pm\sqrt{3}$

Section 2.5 – Zeros of Polynomial Functions

What about when not all the zeros are REAL?

Conjugate Pairs Theorem: Imaginary zeros always come in conjugate pairs.

Corollary to Conjugate Pairs Theorem:
A polynomial of odd degree must have at least one real zero.

Example: Find **all** zeros of the polynomial $f(x) = 3x^4 + 5x^3 + 25x^2 + 45x - 18$

p: ±18, ±9, ±6, ±3, ±2, ±1 → ±18, ±9, ±6, ±3, ±2, ±1, $\pm\frac{2}{3}$, $\pm\frac{1}{3}$
q: ±3, ±1

$$\begin{array}{r|rrrrr} -2 & 3 & 5 & 25 & 45 & -18 \\ & & -6 & 2 & -54 & 18 \\ \hline & 3 & -1 & 27 & -9 & 0 \end{array}$$ Yes!

$(x+2)(3x^3 - x^2 + 27x - 9)$

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & 27 & -9 \\ & & 1 & 0 & 9 \\ \hline & 3 & 0 & 27 & 0 \end{array}$$ Yes!

$(x+2)(x-\frac{1}{3})(3x^2+27)$

Zeros: $-2, \frac{1}{3}, 3i, -3i$

Example: Find **all** zeros of the polynomial $f(x) = x^3 + 13x^2 + 57x + 85$

p: ±85, ±17, ±5, ±1 → ±85, ±17, ±5, ±1
Q: ±1

$$\begin{array}{r|rrrr} -5 & 1 & 13 & 57 & 85 \\ & & -5 & -40 & -85 \\ \hline & 1 & 8 & 17 & 0 \end{array}$$ Yes!

$(x+5)(x^2+8x+17)$

$$\frac{-8 \pm \sqrt{64 - 4(1)(17)}}{2(1)} = -4 \pm \frac{2i}{2}$$
$$= -4 \pm i$$

Zeros: $5, -4+i, -4-i$

Homework: p. 179 #1, 3, 9, 11, 13, 23, 29, 65, 71

Section 2.5 – Zeros of Polynomial Functions

There are a few tricks to help you get a zero with fewer guesses:

Upper and Lower Bounds:

1. If you do synthetic division and get all positive numbers below the line, then the number you used is too high – go lower with your next guess!!
2. If you do synthetic division and get a pattern of positive, negative, positive, negative (or neg-pos-neg-pos) for your numbers below the line, then your guess is too low – go higher with your next guess!!

Descartes' Rule of Signs:

1. If you look at f(x) and count the number of sign changes, that number (possibly less multiples of two) will give you the number of positive real zeros for f(x).
2. If you look at f(-x) and count the number of sign changes, that number (possibly less multiples of two) will give you the number of negative real zeros for f(x).